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Quantum Liouville theory on the pseudosphere with heavy charges [1]

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Abstract

We develop a perturbative expansion of quantum Liouville theory on the pseudosphere around the background generated by heavy charges. Explicit results are presented for the one and two point functions corresponding to the summation of infinite classes of standard perturbative graphs. The results are compared to the one point function and to a special case of the two point function derived by Zamolodchikov and Zamolodchikov in the bootstrap approach, finding complete agreement. A partial summation of the conformal block is also obtained.

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Much interest has been devoted to the exact solutions of the conformal bootstrap equations for the Liouville theory on the pseudosphere [1], on the finite disk with conformally invariant boundary conditions [2] and on the sphere [3, 4, 5]. The first two solutions have given rise to the ZZ and FZZT branes.

In this letter, we present a technique to treat the Liouville field theory on the pseudosphere, which allows to find an expansion in the coupling constant b of the N point functions in presence of "heavy charges", according to the terminology introduced in [3]. This means that we consider the vertex operator $V_\alpha(z) = e^{2\alpha\phi(z)}$ with $\alpha = \eta/b$ and η fixed in the semiclassical limit $b \to 0$.

For the one point function, this analysis goes well beyond the previous perturbative expansion performed in [1, 6, 7] where α has been taken small; indeed, our result corresponds to the summation of an infinite class of perturbative graphs. Thus, we obtain a strong check of the ZZ bootstrap formula for the one point function [1], which includes all the previous perturbative checks.

We apply the same technique to compute the two point function with two heavy charges η and ε, to the first order in ε, getting a closed expression of this correlator to the orders $O(b^{-2})$ and $O(b^0)$ included, but exact in η and in the $SU(1,1)$ invariant distance ω between the sources. According to an argument given by ZZ, such expression provides an expansion of a conformal block up to $O(b^0)$ and to the first order in ε, but to all orders in η and ω. With more work, this technique can be extended to higher orders in b^2 and to more complicated correlation functions.

We start from the Liouville action on the pseudosphere in presence of N sources characterized by heavy charges $\eta_1, \ldots, \eta_N$, given in [7]. Decomposing the Liouville field ϕ as follows

$$\phi \;=\; \frac{1}{2b}\left(\,\varphi_B + 2b\,\chi\,\right) \tag{1}$$

the Liouville action separates into a classical part, depending only on the background field φ_B, and a quantum action for the quantum field χ

$$S_{\Delta,\,N}[\,\phi\,] \;=\; S_{cl}[\,\varphi_B\,] + S_q[\,\varphi_B,\,\chi\,]\;. \tag{2}$$

Adopting the unit disk representation $\Delta = \{z \in \mathbb{C}\,;\,|z| < 1\}$ for the pseudosphere, these

actions read

$$
\begin{aligned}
S_{cl}[\,\varphi_B\,] \;&=\; \frac{1}{b^2}\,\lim_{\substack{\varepsilon\to 0\\ r\to 1}}\;\Bigg\{\,\int_{\Delta_{r,\varepsilon}}\left[\frac{1}{4\pi}\,\partial_z\varphi_B\,\partial_{\bar z}\varphi_B + \mu b^2 e^{\,\varphi_B}\right]d^2z \\
&\qquad -\frac{1}{4\pi i}\,\oint_{\partial\Delta_r}\varphi_B\left(\frac{\bar z}{1-z\bar z}\,dz-\frac{z}{1-z\bar z}\,d\bar z\right)+f_{cl}(r,\mu b^2) \\
&\qquad -\frac{1}{4\pi i}\,\sum_{n=1}^{N}\eta_n\oint_{\partial\gamma_n}\varphi_B\left(\frac{dz}{z-z_n}-\frac{d\bar z}{\bar z-\bar z_n}\right)-\sum_{n=1}^{N}\eta_n^2\log\varepsilon_n^2\;\Bigg\}
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
S_q[\,\varphi_B\,,\chi\,] \;&=\; \lim_{r\to 1}\;\Bigg\{\,\int_{\Delta_r}\left[\frac{1}{\pi}\,\partial_z\chi\,\partial_{\bar z}\chi+\mu\,e^{\,\varphi_B}\left(e^{2b\,\chi}-1-2b\,\chi\right)\right]d^2z \\
&\qquad -\frac{b}{2\pi i}\,\oint_{\partial\Delta_r}\chi\left(\frac{\bar z}{1-z\bar z}\,dz\;-\frac{z}{1-z\bar z}\,d\bar z\right)\;\Bigg\}
\end{aligned}
\tag{4}
$$

where the function $f_{cl}(r,\,\mu b^2)$ is a subtraction term independent of the charges. The coupling constant b is related to the parameter Q occurring in the central charge $c = 1+6Q^2$ by $Q = 1/b+b$ [8]. Moreover, the classical field φ_B obeys the following boundary conditions

$$
\varphi_B(z) \;=\; -\,\log\left(\,1-z\bar z\,\right)^2+f(\mu b^2)+O\big((1-z\bar z)^2\big) \qquad \text{when}\quad |z|\to 1 \tag{5}
$$

$$
\varphi_B(z) \;=\; -\,2\eta_n\log|\,z-z_n\,|^2\;+O(1) \qquad \text{when}\quad z\;\to z_n \tag{6}
$$

where $f(\mu b^2)$ is a function depending only on the product μb^2. The integration domains are $\Delta_{\,r,\,\varepsilon}=\Delta_{\,r}\backslash\bigcup_{n=1}^{N}\gamma_n$ with $\Delta_{\,r}=\{\,|z|<r<1\,\}\subset\Delta$ and $\gamma_n=\{|z-z_n|<\varepsilon_n\}$. Because of the boundary behavior of the Green function, that will be computed in the following, the last integral in the quantum action (4) does not contribute.
The vanishing of the first variation of $S_{cl}[\,\varphi_B\,]$ with respect to the field φ_B satisfying (5) and (6) gives the Liouville equation in presence of N sources

$$
-\,\partial_z\partial_{\bar z}\,\varphi_B\;+2\pi b^2\mu\,e^{\,\varphi_B}\;=\;2\pi\sum_{n=1}^{N}\eta_n\,\delta^2(z-z_n)\;. \tag{7}
$$

At semiclassical level, we have

$$
\langle\,V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\,\rangle_{sc}\;=\;\frac{e^{-S_{cl}(\eta_1\,,\,z_1;\ldots;\,\eta_N\,,\,z_N)}}{e^{-S_{cl}(0)}} \tag{8}
$$

where $S_{cl}(\eta_1,z_1;\ldots;\eta_N,z_N)$ is the classical action $S_{cl}[\,\varphi_B\,]$ computed on the solution φ_B of the Liouville equation with sources. Now, since under a $SU(1,1)$ transformation the

classical background field changes as follows

$$\varphi_B(z) \quad \longrightarrow \quad \tilde{\varphi}_B(w) \;=\; \varphi_B(z) \,-\, \log\left|\frac{dw}{dz}\right|^2 \tag{9}$$

one can see that the transformation law of $S_{cl}[\,\varphi_B\,]$ assigns to the vertex operator $V_\alpha(z)$ the semiclassical dimensions $\alpha\,(1/b-\alpha) = \eta\,(1-\eta)/b^2$ [7, 9].

For the one point function, we have a single heavy charge $\eta_1 = \eta$, which can be placed in $z_1 = 0$, and the explicit solution of the Liouville equation is $\varphi_B = \varphi_{cl}$, given by [10]

$$e^{\varphi_{cl}} \;=\; \frac{1}{\pi\mu b^2}\;\frac{(1-2\eta)^2}{\big[\,(z\bar{z})^{\eta} - (z\bar{z})^{1-\eta}\,\big]^2}\;. \tag{10}$$

Local finiteness of the area around the source in the metric $e^{\varphi_{cl}}\,d^2z$ imposes $\eta < 1/2$ [10, 11].
The classical action (3) computed on this background gives the semiclassical one point function

$$\big\langle\, V_{\eta/b}(0)\,\big\rangle_{sc} \;=\; \exp\left\{-\,\frac{1}{b^2}\,\Big(\eta\,\log\big[\,\pi b^2\mu\,\big] + 2\eta + (1-2\eta)\,\log(1-2\eta)\Big)\right\}\;. \tag{11}$$

To go beyond this approximation, we need to find the Green function on the background field given by (10) and, to do this, we employ the method developed in [12]. Thus, we compute the classical background in presence of the charge η in $z_1 = 0$ and of another charge ε in $z_2 = t \in \Delta$ and real by applying first order perturbation theory in ε to the fuchsian equation associated to the Liouville equation, i.e.

$$\frac{d^2 Y_j}{dz^2} + Q(z)\,Y_j \;=\; 0 \qquad\qquad j \;=\; 1,\,2 \tag{12}$$

where $Q(z) \;=\; b^2 T(z)$, being $T(z)$ the holomorphic component of the classical energy momentum tensor of the classical field φ_B. To first order perturbation theory in ε, one writes $Y_j = y_j + \varepsilon\,\delta y_j$ and $Q = Q_0 + \varepsilon\,q$, where the unperturbed quantities are $Q_0(z) = \eta(1-\eta)/z^2$, $y_1(z) = z^{\eta}$ and $y_2(z) = z^{1-\eta}$.
We impose now the Cardy condition [13] and the regularity condition at infinity on the classical energy momentum tensor. One can express them more easily in the upper half plane $\mathbb{H} = \{\,\xi \in \mathbb{C}\,;\,\mathrm{Im}(\xi) > 0\,\}$ representation (related to the unit disk representation through the Cayley transformation $z = (\xi - i)/(\xi + i)$), where they read $\widetilde{Q}(\xi) \;=\; \overline{\widetilde{Q}}(\xi)$ and $\xi^4\,\widetilde{Q}(\xi) \;\sim\; O(1)$ when $\xi \to \infty$, respectively.
In the $\mathbb{H}$ representation, we have that $\widetilde{Q}_0(\xi) = 4\eta(\eta-1)/(\xi^2+1)^2$ and

$$\tilde{q}(\xi) \;=\; \frac{1}{(\,\xi - i\,\tau\,)^2} + \frac{1}{(\,\xi + i\,\tau\,)^2} + \frac{\beta_i}{2\,(\,\xi - i\,)} + \frac{\bar{\beta}_i}{2\,(\,\xi + i\,)} + \frac{\beta_{i\tau}}{2\,(\,\xi - i\,\tau\,)} + \frac{\bar{\beta}_{i\tau}}{2\,(\,\xi + i\,\tau\,)} \tag{13}$$

where β_i and $\beta_{i\tau}$ are the Poincaré accessory parameters and $i\tau = i(1+t)/(1-t)$ is the image of $z_2 = t$ through the Cayley transformation.

The regularity condition at infinity gives the relations $\mathrm{Re}(\beta_i) = \mathrm{Re}(\beta_{i\tau}) = 0$ and $\mathrm{Im}(\beta_i) = 2 - \tau\,\mathrm{Im}(\beta_{i\tau}) \equiv \beta$, leaving only the parameter β undetermined. Its value is fixed by imposing the monodromy condition of the classical field at 0 and $i\tau$. The result is

$$\beta \;=\; -2\,\frac{\eta + (\,1-\eta\,)\,t^2 - t^{2\,(1-2\eta)}\,(\,1-\eta+\eta\,t^2\,)}{t\,(\,1-t^{2\,(1-2\eta)}\,)}\;. \tag{14}$$

Writing the classical field in presence of two sources of charges η and ε as an expansion up to the first order in ε, i.e.

$$\varphi_2(z) \;=\; \varphi_{cl}(z) + \epsilon\,\psi(z,t) + O(\epsilon^2) \tag{15}$$

one finds that this analysis leads to the following expression for $\psi(z,t)$

$$\begin{aligned}\psi(z,t) \;&=\; -\frac{2}{w_{12}\,(y_1\bar{y}_1 - y_2\bar{y}_2)}\Big\{\,(\,y_1\bar{y}_1 + y_2\bar{y}_2\,)\,(\,I_{12} + \bar{I}_{12} + 2\,h_0\,) \\ &\qquad -\;\bar{y}_1 y_2\,I_{11} -\; y_1\bar{y}_2\,I_{22} -\; y_1\bar{y}_2\,\bar{I}_{11} -\; \bar{y}_1 y_2\,\bar{I}_{22}\Big\}\end{aligned} \tag{16}$$

where $w_{12} = y_1 y_2' - y_1' y_2 = 1 - 2\eta$ is the constant wronskian,

$$I_{ij}(z) \;\equiv\; \int_0^z y_i(x)\,y_j(x)\,q(x)\,dx \tag{17}$$

and h_0 is a free real parameter which cannot be determined through monodromy arguments because it is the coefficient of a solution of the homogeneous equation. It is fixed by requiring the vanishing of $\psi(z,t)$ at infinity, i.e. when $|z| \to 1$, in order to respect the boundary condition (5). The result is

$$h_0 \;=\; \frac{1}{2}\left(\frac{1+t^{2\,(1-2\eta)}}{1-t^{2\,(1-2\eta)}}\;\log t^{2\,(1-2\eta)} + 2\right)\;. \tag{18}$$

Given the expansion (15) and the Liouville equation for $\varphi_2(z)$, it is easy to see that the Green function on the background $\varphi_{cl}(z)$ given by (10) is $g(z,t) = \psi(z,t)/4$. By exploiting the invariance under rotation, we can write our result for a generic complex $t \in \Delta$. The final expression of the exact Green function in the explicit symmetric form is

$$\begin{aligned}g(z,t) \;&=\; -\frac{1}{2}\,\frac{1+(z\bar{z})^{1-2\eta}}{1-(z\bar{z})^{1-2\eta}}\,\frac{1+(t\bar{t})^{1-2\eta}}{1-(t\bar{t})^{1-2\eta}}\,\log\omega(z,t) \;-\; \frac{1}{1-2\eta} \\ &-\frac{1}{1-(z\bar{z})^{1-2\eta}}\,\frac{1}{1-(t\bar{t})^{1-2\eta}}\,\Big\{(z\bar{t})^{1-2\eta}\Big(B_{z/t}(\,2\eta,\,0\,) - B_{z\bar{t}}\,(\,2\eta,\,0\,)\Big) \\ &\qquad + (\bar{z}t)^{1-2\eta}\Big(B_{t/z}(\,2\eta,\,0\,) - B_{1/(z\bar{t})}\,(\,2\eta,\,0\,)\Big) + \text{c.c.}\Big\}\end{aligned} \tag{19}$$

where $\omega(z,t)$ is the $SU(1,1)$ invariant ratio

$$\omega(z,t) \;=\; \frac{(z-t)\,(\bar{z}-\bar{t}\,)}{(1-z\bar{t}\,)\,(1-\bar{z}t)} \tag{20}$$

and $B_x(a,0)$ is a particular case of the incomplete Beta function $B_x(a,b)$ [14]

$$B_x(a,0) \;=\; \frac{x^a}{a}\,F(a,1;a+1;\,x) \;=\; \int_0^x \frac{y^{a-1}}{1-y}\,dy \;=\; \sum_{n=0}^{+\infty}\frac{x^{a+n}}{a+n}\;. \tag{21}$$

From the expression (19), one can verify that $g(z,t) = O\big((1-z\bar{z})^2\big)$ when $|z|\to 1$ [15]. Moreover, in the limit $\eta\to 0$, we recover the propagator $\mathrm{g}(z,t)$ on the pseudosphere without sources given in [16, 1], which has the same boundary behavior at infinity and which has been used to perform the previous perturbative checks of the bootstrap formula for the one point function [1, 6, 7].

A related function playing a crucial role in the following is the Green function at coincident points regularized according to the ZZ procedure [1], i.e.

$$g(z,z) \;\equiv\; \lim_{t\to z}\left\{\,g(z,t)+\frac{1}{2}\,\log|\,z-t\,|^2\,\right\}\;. \tag{22}$$

From (19), we find

$$\begin{aligned} g(z,z) \;=\; & \left(\frac{1+(z\bar{z})^{1-2\eta}}{1-(z\bar{z})^{1-2\eta}}\right)^2 \log\big(\,1-z\bar{z}\,\big) \;-\; \frac{1}{1-2\eta}\,\frac{1+(z\bar{z})^{1-2\eta}}{1-(z\bar{z})^{1-2\eta}} \\ & + \frac{2\,(z\bar{z})^{1-2\eta}}{\big(\,1-(z\bar{z})^{1-2\eta}\,\big)^2}\,\Big(B_{z\bar{z}}\big(\,2\eta\,,0\,\big)+B_{z\bar{z}}\big(\,2-2\eta\,,0\,\big) \\ & \qquad + 2\gamma_E+\psi(2\eta)+\psi(2-2\eta)-\log z\bar{z}\,\Big) \end{aligned} \tag{23}$$

where γ_E is the Euler constant and $\psi(x)=\Gamma'(x)/\Gamma(x)$. When $|z|\to 1$, $g(z,z)$ diverges logarithmically.

The $O(b^0)$ quantum correction to the N point function is given by the quantum determinant $\mathrm{Det}\,D \;=\; \mathrm{Det}\,D(\eta_1,z_1;\ldots;\eta_N,z_N)$

$$\big(\,\mathrm{Det}\,D\,\big)^{-1/2} \;\equiv\; \int\mathcal{D}\,[\,\chi\,]\;\;\exp\left\{-\,\frac{1}{2}\int_\Delta \chi\left(-\,\frac{2}{\pi}\,\partial_z\partial_{\bar{z}}\;+\;4\mu b^2\,e^{\varphi_B}\right)\chi\,d^2z\,\right\} \tag{24}$$

where φ_B is the classical background field solving the Liouville equation with N sources. The quantum determinant can be computed by taking the logarithmic derivative w.r.t. η_j, with $j=1,\ldots,N$, and then integrating back. The key formula is

$$\frac{\partial}{\partial\eta_j}\,\log\big(\,\mathrm{Det}\,D\,\big)^{-1/2} \;=\; -\,2\mu b^2\int_\Delta g(z,z)\,\frac{\partial e^{\varphi_B}}{\partial\eta_j}\,d^2z \qquad\quad \forall\, j\;=\;1,\ldots,\,N \tag{25}$$

which is a convergent integral. From this expression, the transformation laws (9) and

$$g(z,z) \quad\longrightarrow\quad \tilde{g}(w,w) \;=\; g(z,z) + \frac{1}{2}\,\log\left|\,\frac{dw}{dz}\,\right|^2 \tag{26}$$

we can see that the semiclassical dimensions of the vertex operators $V_\alpha(z)$ get corrected to $\Delta_\alpha = \alpha(1/b + b - \alpha)$, which are the quantum dimensions obtained from the hamiltonian approach on the sphere [8]. It is also possible to prove that higher orders corrections do not change these dimensions [15].

In the case of the one point function, the expression (25) can be explicitly computed and the result is

$$\frac{\partial}{\partial\eta}\,\log\big(\,\mathrm{Det}\,D(\eta,0)\,\big)^{-1/2} \;=\; 2\,\gamma_{_E} \;+\; 2\,\psi(1-2\eta) \;+\; \frac{3}{1-2\eta}\;. \tag{27}$$

Integrating back in η with the initial condition given in [1], i.e. $\big(\,\mathrm{Det}\,D(\eta,0)\,\big)^{-1/2}\Big|_{\eta=0} = 1$, we find

$$\log\big(\,\mathrm{Det}\,D(\eta,0)\,\big)^{-1/2} \;=\; 2\,\gamma_{_E}\,\eta - \log\Gamma(1-2\eta) - \frac{3}{2}\,\log(1-2\eta)\;. \tag{28}$$

Putting this result together with the classical contribution (11), we have

$$\begin{aligned}\log\big\langle\, V_{\eta/b}(0)\,\big\rangle \;&=\; -\frac{1}{b^2}\,\Big(\eta\,\log\big[\,\pi b^2\mu\,\big] + 2\eta + (1-2\eta)\,\log(1-2\eta)\Big)\\ &\quad + \Big(2\,\gamma_{_E}\,\eta - \log\Gamma(1-2\eta) - \frac{3}{2}\,\log(1-2\eta)\Big) + O(b^2)\end{aligned} \tag{29}$$

to all orders in η.

We can compare (29) with the result obtained by ZZ within the bootstrap approach [1]. Conformal invariance imposes the following structure for the one point function

$$\langle\, V_\alpha(z_1)\,\rangle = \frac{U(\alpha)}{(\,1 - z_1\bar{z}_1\,)^{\,2\alpha(Q-\alpha)}} \tag{30}$$

where $U(\alpha)$ is the one point structure constant. $U(\alpha)$ has been determined through the bootstrap method [1] and the result for the basic vacuum is

$$U(\alpha) \;=\; U_{1,1}(\alpha) \;=\; \big(\,\pi\mu\gamma(b^2)\,\big)^{-\alpha/b}\,\frac{\Gamma(Q\,b)\,\Gamma(Q/b)\,Q}{\Gamma\big((Q-2\alpha)\,b\big)\,\Gamma\big((Q-2\alpha)/b\big)\,(Q-2\alpha)} \tag{31}$$

where $\gamma(x) = \Gamma(x)/\,\Gamma(1-x)$.

Our result (29) agrees with the expansion in b^2 of $U(\eta/b)$. We stress that (29) corresponds to the summation of an infinite class of graphs of the usual perturbative expansion [1, 6, 7].

The technique developed above can be applied to compute the two point function with one arbitrary heavy charge η and another heavy charge ε

$$\left\langle V_{\eta/b}(0)\, V_{\varepsilon/b}(t) \right\rangle \tag{32}$$

up to the order $O(\varepsilon)$ and $O(b^0)$ included, but to all orders in η and t.
First we observe that it is more convenient to work with the following ratio

$$g_{\eta/b,\,\varepsilon/b}\big(\,\omega(0,t)\,\big) \;\equiv\; \frac{\left\langle V_{\eta/b}(0)\, V_{\varepsilon/b}(t) \right\rangle}{\left\langle V_{\eta/b}(0) \right\rangle \left\langle V_{\varepsilon/b}(t) \right\rangle} \tag{33}$$

where $\omega(z,t)$ is the $SU(1,1)$ invariant ratio (20). Then, the two point function up to the orders $O(\varepsilon)$ and $O(b^0)$ is given by

$$\begin{aligned} \left\langle V_{\eta/b}(0)\, V_{\varepsilon/b}(t) \right\rangle \;&=\; e^{-S_{cl}(\eta,\,0;\,\varepsilon,\,t)}\,\big(\,\mathrm{Det}\,D(\eta,\,0)\,\big)^{-1/2}\;\times \\ &\times\;\left(1\,-\,8\,\mu b^2\varepsilon \int_\Delta g(z,t)\,e^{\varphi_{cl}(z)} g(z,z)\,d^2z \,+O(\varepsilon^2)\right)\left(1+O(b^2)\right) \end{aligned} \tag{34}$$

where $S_{cl}(\eta,0;\varepsilon,t)$ is the classical action (3) with $N=2$ evaluated on the classical field $\varphi_2(z)$, given in (15), which describes the pseudosphere with a curvature singularity of charge η in $z_1=0$ and another curvature singularity of charge ε in $z_2=t$, up to $O(\varepsilon^2)$.
For the logarithm of the ratio (33), we find

$$\begin{aligned} \log\frac{\left\langle V_{\eta/b}(0)\, V_{\varepsilon/b}(t) \right\rangle}{\left\langle V_{\eta/b}(0) \right\rangle \left\langle V_{\varepsilon/b}(t) \right\rangle} \;&=\; \frac{\varepsilon}{b^2}\left\{\,\varphi_{cl}(t) - \varphi_{cl}(t)\big|_{\,\eta=0}\right\} \\ &-8\,\mu b^2\varepsilon\left\{\int_\Delta g(z,t)\,e^{\varphi_{cl}(z)} g(z,z)\,d^2z \,-\int_\Delta \mathrm{g}(z,t)\,e^{\,\varphi_{cl}(t)|_{\,\eta=0}}\,\mathrm{g}(z,z)\,d^2z\right\}\,. \end{aligned} \tag{35}$$

It is possible to show that the r.h.s. of (35) is invariant under $SU(1,1)$ transformations, as expected, and both the integrals occurring in (35) can be explicitly computed. The final result is

$$\begin{aligned} \log\frac{\left\langle V_{\eta/b}(z)\, V_{\varepsilon/b}(t) \right\rangle}{\left\langle V_{\eta/b}(z) \right\rangle \left\langle V_{\varepsilon/b}(t) \right\rangle} \;&=\; \frac{\varepsilon}{b^2}\left\{-\log\frac{\big(\,\omega^\eta-\omega^{1-\eta}\,\big)^2}{(1-2\eta)^2} \,+\,\log(1-\omega)^2\right\} \\ +\,\varepsilon\,\Bigg\{\frac{2}{\big(1-\omega^{1-2\eta}\big)^2}&\Bigg(B_\omega(2-2\eta,0)+\psi(2-2\eta)+\gamma_{\scriptscriptstyle E}+\frac{1}{2(1-2\eta)} \\ &+\,\omega^{2(1-2\eta)}\left(B_\omega(2\eta,0)+\psi(2\eta)+\gamma_{\scriptscriptstyle E}+\frac{3}{2(1-2\eta)}\,-\log\omega\right) \\ &+2\,\omega^{1-2\eta}\left(\log(1-\omega)-\frac{1}{1-2\eta}\right)\Bigg)\,+\,2\,\log(1-\omega)\,-\,3\Bigg\} \end{aligned} \tag{36}$$

where $\omega = \omega(z,t)$ is given by (20).
This expression satisfy the cluster property [1], i.e. $g_{\eta/b,\,\varepsilon/b}(\omega) \to 1$ when $\omega \to 1$.
Setting $\alpha_1 = \eta/b$ and $\alpha_2 = \varepsilon/b$, we have checked that (36) agrees with the standard perturbative expansion up to $O(\alpha_1\alpha_2\, b^2)$ included [7]. A second check consists in the comparison of (36) with the degenerate two point function with $\alpha_1 = -1/(2b)$, i.e. $\eta = -1/2$, and $\alpha_2 = \varepsilon/b$. In this case the ratio (33) is explicitly known [1, 2, 17]

$$g_{-1/(2b),\,\varepsilon/b}(\omega) \;=\; \omega^{\varepsilon/b^2}\,{}_2F_1\big(\,1+1/b^2,\, 2\,\varepsilon/b^2;\, 2+2/b^2;\, 1-\omega\,\big) \tag{37}$$

and we find that the expansion of the logarithm of (37) up to $O(\varepsilon)$ and $O(b^0)$ included reproduces (36) computed for $\eta = -1/2$.
In [1], ZZ gave the ratio (33) in terms of a conformal block with null intermediate dimension as follows

$$g_{\alpha_1,\,\alpha_2}(\omega) \;=\; (1-\omega)^{2\Delta_{\alpha_1}}\,\mathcal{F}\left(\begin{array}{cc}\alpha_1 & \alpha_2\\ \alpha_1 & \alpha_2\end{array};\, i\,Q/2,\, 1-\omega\right)\;. \tag{38}$$

According to this statement, (36) provides an expansion up to $O(\varepsilon)$ and $O(b^0)$, but to all orders in η and ω, of the conformal block (38) with $\alpha_1 = \eta/b$ and $\alpha_2 = \varepsilon/b$.
An extension of the described technique to boundary Liouville field theory [2] is underway [15].

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